Solution of the Four Dimensional Fokker-Planck Equation: Still a Challenge

A. Masud
University of Illinois at Chicago

L.A. Bergman
University of Illinois at Urbana-Champaign

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ABSTRACT: The accurate prediction of the response of a dynamical system is a necessary first step toward its design and eventual control. Development of high fidelity models of the system and excitation processes provides the means to accurately determine system response, assess the adequacy of the system in terms of its performance and safety, and suggest remedial action. However, a degree of uncertainty exists in the properties of nearly every physical system, as well as in most excitations. Thus, the predicted responses, at some scale, will contain some randomness, and the complete and accurate determination of these responses is a nontrivial matter. Here, we discuss the problems inherent in the analysis of a particular class of systems using the finite element method to solve related Fokker-Planck equations.

1 BACKGROUND

Dynamical systems are often subjected to excitations that are most appropriately characterized as random processes. Furthermore, these systems generally exhibit nonlinearity, whether geometric, material, or some combination. As is well known, the determination of response and reliability of even the simplest nonlinear stochastic systems of scientific interest can be a formidable analytical task.

Characterization of the response of randomly excited systems has long interested researchers. For linear systems subjected to additive Gaussian white or linearly filtered Gaussian white noise, the response is known to be Gaussian, and its evolution for a simple linear oscillator, in the context of the Fokker-Planck equation, was first discussed by Wang and Uhlenbeck (1945). Techniques for formulating and, in restrictive cases, solving the Fokker-Planck equation have been discussed by, for example, van Kampen (1981) and Risken (1989, 1996). The generalization to $n$-dimensions can now be found in standard engineering texts such as Lin (1967) and Nigam (1983). For nonlinear systems subjected to additive Gaussian white noise excitations, exact solutions for the transient response process exist only for certain scalar systems as reported, for example, by Caughey and Dienes (1962). However, as discussed by Caughey (1963, 1971), the response of a multidimensional memoryless nonlinear system subjected to additive Gaussian white noise forms a vector Markov process, with transition probability density function satisfying both the forward (Fokker–Planck) and backward Kolmogorov equations. However, these equations remain unsolved except for the linear system of arbitrary dimension and certain scalar nonlinear systems.

The long time, or stationary, response of certain multidimensional nonlinear systems has been obtained through solution of the steady-state Fokker–Planck equation as reported, for example, by Lin and Cai (1988), Soize (1988, 1994). However, these solutions are restrictive, and a vast body of approximation theory, through which the stationary response of more general nonlinear systems can be estimated, has been developed as a result. In one class of approximations, the Fokker–Planck equation is solved numerically for the transition probability density function. Atkinson (1973) examined the stationary response process of several second-order nonlinear systems through solution of the Fokker–Planck equation via Galerkin’s method. Trial functions were the eigenfunctions arising from the Fokker–Planck equation for the linear second order system. Similar methods employing polynomial trial
functions were employed by Wen (1975, 1976) for two and three dimensional nonlinear problems. Similar methods were applied to the stationary Fokker-Planck equation in four dimensions by Wedig and von Wagner (1999), von Wagner and Wedig (2000).

The stationary Fokker–Planck equation for several second-order nonlinear systems was solved using the finite element method by Langley (1985) and Langtangen (1991). Here, the domain of solution was divided into a regular mesh of rectangular elements, and shape functions of class \( C^0 \) were applied to the weak form of the differential equation to develop elemental equations. A difficulty here is satisfying the global normalization condition, handled formally in the latter paper via Lagrange multipliers.

Over the past twenty years, a Petrov-Galerkin finite element method has been used to solve the backward Kolmogorov equation with absorbing boundaries to obtain the solution of the first passage problem for a variety of two and three dimensional dynamical systems incorporating nonlinearity as well as multiplicative excitation (see, for example, Bergman (1989)). Here the transient equations were successfully solved, giving the probability functions of first passage time as well as statistics of the failure process.

More recently, the two dimensional Fokker–Planck equation for several dynamical systems, including the linear, Duffing and van der Pol oscillators, subjected to additive white noise excitation, was solved by Bergman and Spencer (1991), Spencer and Bergman (1993), and Bergman et al. (1996). Here, the Bubnov-Galerkin finite element method was utilized, and the overall size of the problem (10,000 equations; bandwidth of 200) permitted the use of direct solution methods. This work was extended by Wojtkiewicz et al. (1995) to three dimensional problems and by Bergman and Wojtkiewicz (2001) and Wojtkiewicz et al. (2001b, 2001c) to four dimensional problems using both finite element and finite difference formulations, with mixed results, for reasons to be explained shortly.

2 WHERE ARE WE?

The Fokker-Planck, or forward Kolmogorov, equation corresponding to an n-dimensional dynamical system subjected to Gaussian white noise excitation is given by

\[
\frac{\partial f}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j} (vf_j) + \frac{1}{2} \sum_{i,j} \sum_{k,l} \frac{\partial^2}{\partial x_i \partial x_j} \left[ H_{ij}(\nabla f) \right] \tag{1}
\]

where \( f(x,t | x_0,t_0) \) is the transition probability density function, vector \( x \) represents an 'n' dimensional space, and \( z(x) \in \mathbb{R}^n \), \( \mathbf{H}(x) \in \mathbb{R}^n \times \mathbb{R}^n \) are the drift vector and diffusion matrix, respectively. The initial condition is given by

\[
f(x,0 | x_0) = \prod_{i=1}^{n} \delta(x_i - x_{i0}) \tag{2}
\]

We first define the following operators:

\[
\nabla = \text{gradient operator in 'n' dimensions}, \quad \nabla \cdot = \text{divergence operator in 'n' dimensions}, \quad \Delta = \text{Laplace operator in 'n' dimensions}.
\]

The Fokker-Planck equation can be written in direct notation as

\[
f' = -\nabla \cdot (zf) + \frac{1}{2} \Delta (\mathbf{H}f), \quad \Omega \times [0,T] \tag{3}
\]

This equation, together with the initial condition, constitutes the strong form of the problem. Now, let \( w \in H^1(\Omega) \) be the weighting function for the probability density function. Taking the inner product of the weighting function with the previous equation and applying the divergence theorem yields the weak form of the problem

\[
\left( w, f' \right) + \left( w, \nabla \cdot (zf) \right) + \frac{1}{2} \left( \nabla w, \mathbf{H} \nabla f \right) = 0 \tag{4}
\]

where \( \left( \cdot, \cdot \right) \) is the \( L_2 \) inner product. In the final step, we let \( \psi \in H^1(\Omega) \cap C^0(\Omega) \) represent the space of piecewise continuous trial solutions and weighting functions. Substituting the discrete counterparts of \( f \) and \( w \) into the last equation yields the standard Galerkin form of the problem.

Currently, solving Fokker-Planck equations in two spatial dimensions by the finite element method has been reduced to the level of a web-based demonstration (Wojtkiewicz et al. 2001a). In most cases, a uniform mesh of \( 100 \times 100 \) 4-node rectangular elements of class \( C^0 \) over the phase plane provides an upper bound for highly accurate solution (approximately 10,000 linear equations), assuming the extent of the boundaries is consistent with the initial conditions of the problem, thus avoiding reflections from the boundaries back into the interior. All of the necessary algorithms for
discretization and solution are widely available and understood, and existing finite element codes capable of solving advective-diffusion problems are easily adapted to the Fokker-Planck equation. In fact, if neither the system nor the mesh is time-varying and the integration time step is constant, just a single reduction of the equations to upper-triangular form following discretization followed by repetitive reformulation of the right-hand-side and back-substitution, yields the solution to stationarity in a few seconds on a laptop computer.

Simple verification of this is afforded by the solution of the Fokker-Planck equation for the two-dimensional linear oscillator subjected to additive white noise excitation, for which an exact solution exists. The non-dimensionalized Fokker-Planck equation is given by

\[
\frac{1}{2\pi} \frac{\partial f}{\partial \tau} = 2\xi \frac{\partial^2 f}{\partial y^2} - \frac{\partial}{\partial y}(2\xi y + x)f - \frac{\partial}{\partial x}(yf) \tag{5}
\]

Here, \( x = x_1/\sigma \), \( y = x_2/\sigma \omega_1 \), \( \tau = \omega_0 t/2\pi \), and \( \sigma^2 = \pi S_1 / 2\xi \omega_1 \). The damping parameter is \( \xi = 0.2 \). The computational domain is given by \(-10 \leq x, y \leq 10\), where 10 is chosen as the extent of the mesh, based on experience, as necessary in transient computations with non-zero initial conditions to minimize the incidence of probability mass reflecting off the boundaries and back into the interior. The mesh employed in the simulation pictured is composed of \( 50 \times 50 \) 4-node elements, and the time-integration method is the Crank-Nicholson algorithm with \( \Delta t = 0.005 \). Boundary conditions are set such that \( f = 0 \) at the boundaries. The initial condition is a Gaussian distribution with standard deviation of unity, centered at \((x, y) = (-3, -3)\).

Figure 1 shows the evolution of the probability density function at \((x, y) = (0, 0)\). Note that, at stationarity, the value of the distribution at the origin converges to \( f(0,0) = 1/2\pi \), the exact solution. Figure 2-a shows the exact probability density function at stationarity, while Figure 2-b shows the corresponding computational result. There is no difference between the two to machine precision.

The situation for three-state problems, typically a single degree of freedom system with a first order pre-filter or a single degree of freedom system with a nonlinear constitutive law requiring an additional internal state variable, is much the same. Here, a uniform mesh of \( 100 \times 100 \times 100 \) 8-node brick elements of class \( C^0 \) over the phase space provides an upper bound for highly accurate solution (approximately 1,000,000 linear equations). Existing finite element codes capable of solving advective-diffusion problems in three dimensions are readily adapted to this problem,
and as long as neither the system nor the mesh is time-varying and the integration time step is constant, a single reduction of the equations to upper-triangular form, followed by repetitive reformulation of the right-hand-side and back-substitution, yields the solution to stationarity in a few minutes on a laptop computer.

A three state linear stochastic dynamical system first reported in Wojtkiewicz et al. (1995) serves as a benchmark to assess the accuracy and stability of the developed formulation. The equations of motion are

\[
\begin{align*}
\dot{x}_1 &= -\gamma x_1 - 2\xi x_2 - \varepsilon x_3 + w(t), \\
\dot{x}_2 &= -\gamma x_2 - 2\xi x_3 + x_1 + 0, \\
\dot{x}_3 &= -\alpha x_3 + 1,
\end{align*}
\]

where \( w(t) \) is a Gaussian white noise process with

\[
E[w(t)] = 0, \quad E[w(t)w(t')] = 2D\delta(t - t')
\]

for \( t = t_1 - t_2 \). For this case, an exact stationary solution of the Fokker-Planck equation exists and is given by

\[
f(x) = \left((2\pi)^3 |\Gamma|\right)^{-1/2} \exp\left(-\frac{1}{2} x^T \Gamma^{-1} x\right)
\]

where \( \Gamma \) is the stationary covariance matrix of the response. Parameters employed are: \( \varepsilon = 1, \gamma = \pm 1, \omega_0 = 1, \xi = 0.2, \alpha = 1, D = 0.4 \).

The computational domain is \(-4 \leq x_1, x_2, x_3 \leq 4\). The relatively coarse uniform mesh employed in the simulation is composed of \(24 \times 24 \times 24\) 8-node brick elements, and the time integration scheme is the Crank-Nicholson algorithm with \( \Delta t = 0.01 \).

The boundary conditions employed here are \( f = 0 \) on all of the boundaries.

Two problems are considered. In the first, the initial condition is a Gaussian distribution with standard deviation of unity and variance of 0.2, centered at \((x_1, x_2, x_3) = (0, 0, 0)\). Figure 3-a shows the evolution of the probability density function \( f \) at \((0, 0, 0)\). In the second case the initial condition is a Gaussian distribution with standard deviation of unity and initial variance 0.2, centered at \((x_1, x_2, x_3) = (-1.5, -1.5, -1.5)\). Figure 3-b shows the corresponding evolution of the probability density function \( f \) at \((0, 0, 0)\). In both cases the exact solution at stationarity is 0.241. The observed error is attributable to the coarseness of the mesh used in the discretization.

![Figure 3. Probability density function for the linear oscillator at \((x_1, x_2, x_3) = (0, 0, 0)\): (a) initial distribution centered at \((0,0,0)\); (b) initial distribution centered at \((-1.5, -1.5, -1.5)\).](image)

3 WHY IS IT DIFFICULT TO PROGRESS TO HIGHER DIMENSIONS?

As noted earlier, there has been some progress toward solution of four dimensional problems by both finite element and finite difference methods, reported in Bergman and Wojtkiewicz (2001), Wojtkiewicz et al. (2001b, 2001c). However, these early attempts, and further extensions, have proved to be problematic. First of all, beyond three dimensions we no longer have the luxury of utilizing existing codes. Indeed, new codes must be constructed with entirely new data structures that will efficiently accommodate increasingly larger dimensionality. Second, continued use of dense, time-invariant, uniform meshes in problems of increasingly large dimension results in systems of equations that quickly approach and exceed the limits of current computational platforms. Finally,
higher spatial dimensionality requires greater resolution in the time integration scheme, particularly at early times in the simulation, in order to accurately resolve features in the rapidly evolving probability flow. This leads to vanishingly small time steps over the entire period of the solution, unless intermediate reformulations are considered, resulting in intractable solution times.

Thus, it seems clear that these problems must be approached differently, in a way that enables efficient solution of the Fokker-Planck equation for reasonably large dimensionality.

4 A ROADMAP TO SOLUTION OF HIGHER DIMENSIONAL PROBLEMS

Efficient solution of the Fokker-Planck equation in higher dimensions will require several parallel developments. These will make possible the solution, in the near term, of systems of several degrees of freedom and, in the long term, of systems with many degrees of freedom, encompassing a large number of fundamental problems of current interest in stochastic dynamics. Our future efforts in support of this goal will have three thrusts:

1. Development of multi-scale finite element methods for the solution of the multi-dimensional Fokker-Planck equation. Here, we seek to develop finite element methods that yield higher accuracy on cruder meshes. This would reduce the size of the computational grid, thus reducing the computational overhead associated with large system of equations that arise in higher dimensions. A framework for this procedure has already been discussed in Masud and Bergman (2004a, 2004b).

2. Development of dynamic grid methods with moving grid points to reduce the size of the computational problem. In the absence of dynamic moving grids, problems with changing features, typical of Fokker-Planck equation with non-zero initial conditions, require uniform mesh refinement over the entire domain. In a dynamic grid scheme, the grid refinement follows the regions of peak activity by coarsening the region where the solution is very smooth and relocating the grid points to regions with high gradients. This considerably reduces

3. Development of higher-order accurate time integration schemes on moving grids. For solutions of the Fokker-Planck equation on fixed meshes, we have utilized the Crank-Nicholson method, which is an unconditionally stable, second order-accurate numerical scheme. We have shown that, if the Crank-Nicholson method is employed for calculations on dynamic meshes, the weak instabilities that are triggered because of mesh motion can not be damped out. Although there exist higher order-accurate time integration schemes for systems of first order equations, these schemes require data from several previous time steps in addition to the current one. For the moving mesh problem, since the computational mesh moves and changes within the time step, transferring the data from previous meshes necessarily involves projection methods. Designing second order projection operators can be as expensive as the time integration scheme itself. In view of these limitations, it is imperative to design smart time integration schemes for moving meshes that are one-step methods, second order accurate, and unconditionally stable.

Although the moving mesh will reduce the size of the computational problem, it will require the governing equation to be written in an Arbitrary-Lagrangian-Eulerian (ALE) framework. This will necessitate a complete stability and convergence analysis in a general ‘n’ dimensional space in order to effectively assess the efficacy of the code.

5 CONCLUSIONS

We have provided a short summary of progress made, as well as a likely roadmap to future developments, that will enable the integration of Fokker-Planck equations in dimension four and, possibly, greater within the limitations of current hardware. Implementation will largely depend on the availability of funding to support the effort. The authors are hopeful that will be secured in the relatively near term.
DEDICATION

The authors dedicate this paper to the memory of Professor Thomas F. Caughey.

REFERENCES


